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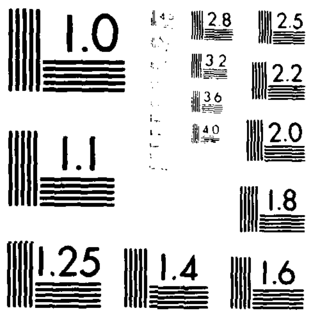

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CONSTITUTIVE MODEL FOR RAPIDLY DAMAGED STRUCTURAL MATERIALS. II. FINITE ELEMENT FORMULATION

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25 JULY 1980

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A constitutive model was recently proposed to describe flow and damage in rapidly loaded structural materials. Its finite element implementation is given in the present work, with numerical results to be reported in a subsequent work. One important feature is that the finite element formulation is "consistent" in that the damage and flow strains are approximated in the same way as the corresponding parts of the total strain. A second important feature is that certain interelement continuity conditions are imposed in the

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flow and damage strains. A system of ordinary differential equations in time is obtained for the nodal displacement, flow and damage parameters.

FOREWORD

This work reports part of two man-years of effort supported by the Advanced Lightweight Torpedo Project. The object is the direct prediction of the onset and details of submarine pressure hull rupture caused by underwater explosive attack. An earlier report developed a material model describing plasticity and rupture under extremely rapid loading conditions. This work gives its finite element implementation, with numerical results to be communicated in a subsequent report.

ELIHU ZIMET
By direction

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INTRODUCTION

In ductile structural metals such as aluminum, the response to rapidly applied high loads can involve two basic mechanisms [1,2]: (a) flow, understood physically as slip within grains, and (b) damage, comprising the nucleation of microvoids at grain interfaces, their subsequent growth, and their eventual, usually abrupt, coalescence into cracks. Reference 1 introduces a constitutive model extending viscoplasticity to accommodate both flow and damage. The present work concerns the finite element implementation of the model. Numerical results will be presented in a later work.

CONSTITUTIVE MODEL

We briefly state the constitutive relations given in Reference 1 under restriction to small strains and temperature independent deformations.

In obvious notation, the strain is decomposed into elastic, flow and damage parts according to

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{f} + \varepsilon_{ij}^{d} \tag{1}$$

and the elastic strain is given by Hooke's law as

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij}^{e} + \lambda\, \varepsilon_{kk}^{e}\, \delta_{ij} \tag{2}$$

where μ and λ are the Lamé coefficients and δ_{ij} is the Kronecker tensor.

Let e_{ij} and e be the deviatoric (shear) and isotropic (dilatational) parts of the strain tensor, and s_{ij} and s correspondingly for σ_{ij}. In Reference 1, constitutive relations embodying associated flow rules were developed as

1. Nicholson, D. W., "Constitutive Model for Rapidly Damaged Structural Materials," accepted for publication in Acta Mechanica
2. Barbee, T. W., et al, "Dynamic Fracture Criteria for Ductile and Brittle Metals," J. Materials, 1972

$$\dot{e}^f_{ij} = \eta_f < \phi_f \, (F_f - k^f) > \frac{\partial F_f}{\partial s_{ij}} \tag{3a}$$

$$\dot{e}^f = 0 \tag{3b}$$

$$\dot{e}^d = \eta_d < \phi_d \, (F_d - k^d) > \frac{\partial F_d}{\partial s} \tag{3c}$$

$$\dot{e}^d_{ij} = 0. \tag{3d}$$

Here η_f and η_d are material constants, k^f and k^d are parameters representing dependence on the history of flow and damage, ϕ_f, ϕ_d, F_f and F_d are material functions and the symbols $< \cdot >$ are defined by

$$< \psi \, (\Gamma) > = \begin{cases} 0 & , \quad \Gamma \leq 0 \\ \Gamma & , \quad \Gamma > 0 \; . \end{cases}$$

The material function F_f depends on e^f_{ij}, k^f and s_{ij}, while F_d depends on e^d, k^d and s. Finally, the history parameters are governed by

$$\dot{k}^f = h^f_{ij}(e^f_{pq}, \, k^f, \, s_{pq}) \, \dot{e}^f_{ij} \tag{3e}$$

$$\dot{k}^d = h^d \, (e^d, \, k^d, \, s) \, \dot{e}^d \tag{3f}$$

FINITE ELEMENT FORMULATION

A. Equation of Equilibrium for an Element

Suppose that high loads are rapidly applied to a body governed by Equation 3a-f. The body can be represented as a collection of finite elements connected to each other at nodes [3]. We consider equilibrium of a given element.

In accordance with the usual practice in finite element analysis, we hereafter use vector notation. So ε_{ij} is replaced by $\underline{\varepsilon}$, σ_{ij} by $\underline{\sigma}$, etc.

3. Zienkiewicz, O. C., The Finite Element Method, Third Edition, McGraw-Hill Book Co., New York, 1977

Let the vector $\underline{r}$ denote the position of a given interior point of the element under study. The time displacement vector $\underline{u}(\underline{r})$ is approximated by $\overline{\underline{u}}(\underline{r})$ according to

$$\overline{\underline{u}}\ (\underline{r}) = N\ (\underline{r})\ \underline{\zeta} \tag{4}$$

where $\underline{\zeta}$ is the vector of nodal displacements and the matrix N ($\underline{r}$) is an "interpolation operator."

For the sake of illustrating the oftentimes bewildering finite element notation, it is convenient to use the simple triangle element shown in Figure 1. Its i^{th} node is at (x_i, y_i), at which the displacements are $(u_x^{(i)}, u_y^{(i)})$. Now let

$$\underline{r} = \{x\ y\}^H$$

$$\underline{u} = \{u_x\ u_y\}^H$$

$$\underline{\zeta} = \{u_x^{(1)}\ u_y^{(1)} \vdots u_x^{(2)}\ u_y^{(2)} \vdots u_x^{(3)}\ u_y^{(3)}\}^H$$

in which the superscript H denotes the transpose.

For the triangle we assume the linear approximation

$$u_x = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$u_y = \alpha_4 + \alpha_5 x + \alpha_6 y\ .$$

In vector notation

$$\underline{u} = D\ \underline{\alpha}$$

in which $\underline{\alpha}$ is the constant vector

$$\underline{\alpha} = \{\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\}^H$$

and

$$D = \begin{bmatrix} 1 & x & y & o & o & o \\ o & o & o & 1 & x & y \end{bmatrix}.$$

It is elementary to derive that

$$\underline{\bar{u}} = D\ C\ \underline{\zeta}$$

where

$$C = \begin{bmatrix} 1 & o & x_1 & o & y_1 & o \\ o & 1 & o & x_1 & o & y_1 \\ 1 & o & x_2 & o & y_2 & o \\ o & 1 & o & x_2 & o & y_2 \\ 1 & o & x_3 & o & y_3 & o \\ o & 1 & o & x_3 & o & y_3 \end{bmatrix}^{-1}$$

Returning to the general discussion, the true strain $\underline{\varepsilon}$ in an element may be written as

$$\underline{\varepsilon} = B' * \underline{u}$$

where B′ is a kinematic operator. Applying B′ to $\underline{\bar{u}}$ furnishes a strain approximation as

$$\underline{\bar{\varepsilon}} = B' * \underline{\bar{u}}$$

$$= B\ C\ \underline{\zeta} \qquad (5)$$

where $B(\underline{r})$ is a matrix.

For the triangular element we find

$$\underline{\varepsilon} = \{\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{xy}\}^H$$

from which

$$B = \begin{bmatrix} o & 1 & o & o & o & o \\ o & o & o & o & o & 1 \\ o & o & \tfrac{1}{2} & 1 & 1 & \tfrac{1}{2} \end{bmatrix}.$$

The true stress vector $\underline{\sigma}(\underline{r})$ may in general be expressed as a functional

of $\underline{\varepsilon}$, with the actual functional form determined by the constitutive model. Formally,

$$\underline{\sigma}(\underline{r}) = \Lambda(\underline{\varepsilon}, \underline{r}) \tag{6}$$

The approximate stress is obtained from

$$\overline{\underline{\sigma}}(\underline{r}) = \Lambda(\overline{\underline{\varepsilon}}, \underline{r})$$

$$= \Lambda' (\underline{\xi}, \underline{r}) \; .$$

In the triangle, assuming linear isotropic elasticity, it follows that

$$\overline{\underline{\sigma}} = \text{EBC } \underline{\xi}$$

where E is a matrix of elastic constants.

For equilibrium of an element, the principle of virtual work may be stated in terms of true quantities as

$$\int_V \rho \; \underline{\ddot{u}}^H \; \delta \underline{u} dV + \int_V \underline{\sigma}^H \; \delta \underline{\varepsilon} dV = \int_S \underline{\tau}^H \; \delta \underline{u} dV \tag{7}$$

In Equation 7, V is the element volume and S its surface area, τ is the traction applied to the element boundary, ρ is the mass density, $\delta(\cdot)$ is the variational operator, and the superposed dot denotes differentiation with respect to time.

We assume this principle also applies to the approximate quantities:

$$\int_V \rho \; \overline{\underline{\ddot{u}}}^H \; \delta \overline{\underline{u}} dV + \int_V \overline{\underline{\sigma}}^H \; \delta \overline{\underline{\varepsilon}} dV = \int_S \underline{\tau}^H \; \delta \overline{\underline{u}} dS \; . \tag{8}$$

Hereafter, the overbars designating the approximations will not be displayed.

Upon substituting Equation 4, the right hand term in Equation 8 becomes

$$\int_S \underline{\tau}^H \, \delta\underline{\bar{u}} dS = \left[\int_S \underline{\tau}^H \, DCdS \right] \delta\underline{\zeta}$$

$$\equiv P^H \, \delta\underline{\zeta}$$

and $\underline{P}$ may be called the consistent load vector.

For the inertial term

$$\int_V \rho\ddot{\underline{u}}^H \, \delta u dV = \ddot{\underline{\zeta}}^H \left[\int_V \rho C^H D^H DCdV \right] \delta\underline{\zeta}$$

$$= \ddot{\underline{\zeta}}^H \, M \, \delta\underline{\zeta}$$

and M is the consistent mass matrix.

The second term on the left hand side involves the constitutive model. First write $\underline{\sigma}$ and $\underline{\varepsilon}$ in deviatoric and isotropic components as

$$\underline{\sigma} = \underline{s} + s \, \underline{\theta}$$

$$\underline{\varepsilon} = \underline{e} + e \, \underline{\theta}$$

where $\underline{\theta}$ is the vectorial counterpart of the Kronecker tensor. Elementary manipulation leads to

$$\underline{\sigma}^H \, \delta \, \underline{\varepsilon} = \underline{s}^H \, \delta \, \underline{e} + 3 \, s \, \delta \, e \quad .$$

From the constitutive model

$$\underline{s} = 2\mu \, (\underline{e} - \underline{e}^f)$$

$$s = \kappa \, (e - e^d)$$

with $\kappa = 2\mu + 3\lambda$. Consequently,

$$\underline{\sigma}^H \, \delta\underline{\varepsilon} = 2\mu \, e^H \, \delta e + 3\kappa \, e\delta e$$

$$- 2\mu \, e^{f^H} \, \delta e - 3\kappa \, e^d \delta e$$

Equation 5 implies that

$$\underline{e} = B'C\,\underline{\xi} \quad (9a)$$

$$e = \underline{b}^H C\,\underline{\xi} \quad (9b)$$

where the matrix B' and the vector $\underline{b}$ are easily derived when B is specified.

For example, in the triangle element

$$e = (\varepsilon_{xx} + \varepsilon_{yy})/3$$

$$e_{xx} = \varepsilon_{xx} - e \qquad e_{yy} = \varepsilon_{yy} - e$$

$$e_{zz} = -e \qquad e_{xy} = \varepsilon_{xy}$$

and

$$\underline{e} = \{e_{xx}\ e_{yy}\ e_{zz}\ e_{xy}\}^H$$

Simple algebra leads to

$$b = \{o\ 1/3\ o\ o\ o\ 1/3\}^H$$

$$B' = \begin{bmatrix} o & 2/3 & o & o & o & -1/3 \\ o & -1/3 & o & o & o & 2/3 \\ o & -1/3 & o & o & o & -1/3 \\ o & o & 1/2 & o & 1/2 & o \end{bmatrix}$$

Up to this stage in our development, nothing has been said about the distribution of $\underline{e}^f$ and e^d in an element. We now assume that they are distributed in the same way as the corresponding parts of the strain tensor. Formally,

$$\underline{e}^f = B'C\beta \quad (10a)$$

$$e^d = \underline{b}^H\,C\gamma \quad (10b)$$

in terms of new unknown vectors $\underline{\beta}$ and $\underline{\gamma}$, called the flow and damage parameters. The prime in Equation 10a will no longer be displayed. The assumption

expressed in Equations 10a, b leads us to call the present relations a consistent inelastic formulation.

It now follows that

$$\int_V \underline{\sigma}^H \delta \underline{\varepsilon} \, dV = \underline{\zeta}^H K_f \delta\underline{\zeta} + \underline{\zeta}^H K_d \delta\underline{\zeta} - \underline{\beta}^H K_f \delta\underline{\zeta} - \underline{\gamma}^H K_d \delta\underline{\zeta}$$

with

$$K_f = 2\mu \int_V C^H B^H B C dV$$

$$K_d = 3\kappa \int_V C^H b^* b^H C dV$$

and

$$\{b^* b^H\}_{ij} = b_i b_j \quad .$$

The matrix K_e given by

$$K_e = K_f + K_d$$

is nothing but the ordinary stiffness matrix of linear elasticity.

The equilibrium relation for the element under study is now:

$$M \ddot{\underline{\zeta}} + (K_f + K_d)\underline{\zeta} = \underline{P} + K_f \underline{\beta} + K_d \underline{\gamma} \tag{11}$$

In the next section we use the constitutive model to derive equations governing $\underline{\beta}$ and $\underline{\gamma}$. They will have the general form

$$\dot{\underline{\beta}} = \underline{z}^f (\underline{\zeta}, \underline{\beta}, \hat{k}^f) \tag{12a}$$

$$\dot{\hat{k}}^f = w^f (\underline{\zeta}, \underline{\beta}, \hat{k}^f) \tag{12b}$$

$$\dot{\underline{\gamma}} = \underline{z}^d (\underline{\zeta}, \underline{\gamma}, \hat{k}^d) \tag{12c}$$

$$\dot{\hat{k}}^d = w^d (\underline{\zeta}, \underline{\gamma}, \hat{k}^d) \tag{12d}$$

where z^f, w^f, z^d and w^d are material functions. More concretely, in the next section the constitutive model will be used to derive the material functions in Equation 12a-d.

B. Finite Element Form of the Constitutive Equations

The constitutive Equations 3a, d, e, and f may be rewritten as

$$\dot{\underline{e}}^f = \underline{g}^f\ (\underline{s},\ \underline{e}^f,\ k^f) \tag{13a}$$

$$\dot{k}^f = h^f\ (\underline{s},\ \underline{e}^f,\ k^f) \tag{13b}$$

$$\dot{e}^d = g^d\ (s,\ e^d,\ k^d) \tag{13c}$$

$$\dot{k}^d = h^d\ (s,\ e^d,\ k^d) \tag{13d}$$

Our task is now to restate the constitutive relations in terms of $\underline{\zeta}$, $\underline{\beta}$ and $\underline{\gamma}$ and to eliminate dependence on $\underline{r}$.

From the constitutive model we may write

$$\underline{e}^f = BC\ \beta \tag{14a}$$

$$s = 2\mu\ (\underline{e} - \underline{e}^f) = 2\mu\ BC\ (\underline{\zeta} - \underline{\beta}) \tag{14b}$$

$$e^d = b^H C\ \gamma \tag{14c}$$

$$\begin{aligned} s &= k\ (e - e^d) \\ &= k\ b^H C\ (\underline{\zeta} - \underline{\gamma}) \end{aligned} \tag{14d}$$

Recall that B, $\underline{b}$ and C may depend on $\underline{r}$.

Note that k^f and k^d appear in the arguments on the right hand side of Equation 13a-d. We now make the additional assumption that they may be replaced by $\hat{k}^f$ and $\hat{k}^d$ where the circumflexes denote the element volume averages:

$$\hat{k}^f = \frac{1}{V} \int_V k^f\ dV \tag{15a}$$

$$\hat{k}^d = \frac{1}{V} \int_V k^d\ dV \tag{15b}$$

From Equations 13a-d, 14a-d and 15a-b, it is evident that

$$\underline{\dot{e}}^f = \underline{g}^f\,(2\mu BC(\underline{\xi} - \underline{\beta}),\ BC\underline{\beta},\ \hat{k}^f)$$

$$= \underline{q}^f\,(\underline{\xi},\ \underline{\beta},\ \hat{k}^f,\ \underline{r}) \tag{16a}$$

where the function $\underline{q}^f$ is defined in Equation 16a. By similar argument,

$$k^f = p^f\,(\underline{\xi},\ \underline{\beta},\ \hat{k}^f,\ \underline{r}) \tag{16b}$$

$$\dot{e}^d = q^d\,(\underline{\xi},\ \underline{\gamma},\ \hat{k}^d,\ \underline{r}) \tag{16c}$$

$$\dot{k}^d = p^d\,(\underline{\xi},\ \underline{\gamma},\ \hat{k}^d,\ \underline{r}) \tag{16d}$$

We seek to eliminate dependence on $\underline{r}$. But Equation 16a implies that

$$\int_V C^H B^H\,\underline{\dot{e}}^f\,dV = (2\mu)^{-1} K_f\,\dot{\beta}$$

$$= \int_V N^H B^H\,\underline{q}^H\,dV\ .$$

Therefore, the material function $\underline{z}^f$ in Equation 12a is given by

$$\underline{z}^f\,(\underline{\xi},\ \underline{\beta},\ \hat{k}^f) = 2\mu K_f^{-1} \int_V C^H N^H\,\underline{q}^f\,dV \tag{17a}$$

Similar manipulations serve to derive w^f, $\underline{z}^f$, and w^d in Equation 12b-d:

$$w^f = V^{-1} \int_V p^f\,dV \tag{17b}$$

$$\underline{z}^d = 3k\ K_d^{-1} \int_V N^H \underline{b}\ q^d\,dV \tag{17c}$$

$$w^d = V^{-1} \int_V p^d\,dV \tag{17d}$$

For the sake of illustration we consider the constitutive relations

$$\underline{\dot{e}}^f = n_f < 1 - k_f/F_f > (\underline{s} - c_f\,\underline{e}^f) \tag{18a}$$

$$F_f = [(\underline{s} - c_f\, \underline{e}^f)^H (\underline{s} - c_f\, \underline{e}^f)]^{1/2} \tag{18b}$$

$$\dot{e}^d = \eta_d < 1 - k_d/F_d > (s - c_d\, e^d) \tag{18c}$$

$$F_d = (s - c_d\, e^d)^2$$

where k_f, k_d, η_f, η_d, c_f and c_d are material constants. These relations were previously introduced in Reference 1.

Applied to the triangular element, the relations furnish

$$\dot{\underline{\beta}} = \nu_f\ \underline{\Psi}_f \tag{19a}$$

in which

$$\underline{\Psi}_f = 2\,\mu\,\underline{\zeta} - (2\mu + c_f)\,\underline{\beta} \tag{19b}$$

$$\nu_f = (2\mu\, A)^{-1/2}\,[\underline{\Psi}_f^H\, K_f\, \underline{\Psi}_f]^{1/2} \tag{19c}$$

where A is the area of the element.

For damage,

$$\dot{\underline{\gamma}} = \nu_d\, \underline{\Psi}_d \tag{20a}$$

with

$$\underline{\Psi}_d = \kappa\,\underline{\zeta} - (\kappa + c_d)\,\underline{\gamma} \tag{20b}$$

$$\nu_d = (3\kappa A)^{-1/2}\,[\underline{\Psi}_d^H\, K_d\, \underline{\Psi}_d]^{1/2} \tag{20c}$$

C. Nodal Continuity and Equilibrium

The previous section concerned equilibrium of a given element. Here we consider equilibrium and compatibility of the assemblage of elements, for example the triangles shown in Figure 2. For this purpose it is adequate to develop force balance and continuity equations holding at the shared nodes. Certain modifications of the single element relations will prove convenient.

First, some additional notation is needed. The quantities $\underline{\ }^{(e)}$, $\underline{\ }^{(e)}$, $\underline{\zeta}^{(e)}$, $K_f^{(e)}$, $K_d^{(e)}$, $M^{(e)}$, $\hat{k}^f{}_e$ and $\hat{k}^d{}_e$ now refer to the e^{th} element, for example the second element in Figure 2.

Let ${}^{(e)}\zeta_m^{(n)}$ be the entry of $\underline{\ }^{(e)}$ referring to the n^{th} node and the m^{th} direction. For instance, since $\underline{\zeta}^{(e)}$ is the nodal displacement vector for the e^{th} element, then ${}^{(e)}\zeta_x^{(n)}$ is the x-displacement of its node having n as its index. In reference to Figure 2,

$$\underline{\zeta}^{(1)} = \{{}^{(1)}\zeta_x^{(1)} \quad {}^{(1)}\zeta_y^{(1)} \quad {}^{(1)}\zeta_x^{(4)} \quad {}^{(1)}\zeta_y^{(4)} \quad {}^{(1)}\zeta_x^{(5)} \quad {}^{(1)}\zeta_y^{(5)}\}^H$$

$$\underline{\zeta}^{(2)} = \{{}^{(2)}\zeta_x^{(1)} \quad {}^{(2)}\zeta_y^{(1)} \quad {}^{(2)}\zeta_x^{(2)} \quad {}^{(2)}\zeta_y^{(2)} \quad {}^{(2)}\zeta_x^{(4)} \quad {}^{(2)}\zeta_y^{(4)}\}^H$$

$$\underline{\zeta}^{(3)} = \{{}^{(3)}\zeta_x^{(2)} \quad {}^{(3)}\zeta_y^{(2)} \quad {}^{(3)}\zeta_x^{(3)} \quad {}^{(3)}\zeta_y^{(3)} \quad {}^{(3)}\zeta_x^{(4)} \quad {}^{(3)}\zeta_y^{(4)}\}^H .$$

Continuity of displacements implies that

$$\begin{aligned}
&{}^{(1)}\zeta_x^{(1)} = {}^{(2)}\zeta_x^{(1)} = u_x^{(1)} && {}^{(1)}\zeta_y^{(1)} = {}^{(2)}\zeta_y^{(1)} = u_y^{(1)} \\
&{}^{(2)}\zeta_x^{(2)} = {}^{(3)}\zeta_x^{(2)} = u_x^{(2)} && {}^{(2)}\zeta_y^{(2)} = {}^{(3)}\zeta_y^{(2)} = u_y^{(1)} \\
&{}^{(3)}\zeta_x^{(3)} = u_x^{(3)} && {}^{(3)}\zeta_y^{(3)} = u_y^{(3)} \qquad (21) \\
&{}^{(1)}\zeta_x^{(5)} = u_x^{(5)} && {}^{(1)}\zeta_y^{(3)} = u_y^{(5)}
\end{aligned}$$

$${}^{(1)}\zeta_x^{(4)} = {}^{(2)}\zeta_x^{(4)} = {}^{(3)}\zeta_x^{(4)} = u_x^{(4)}$$

$${}^{(1)}\zeta_y^{(4)} = {}^{(2)}\zeta_y^{(4)} = {}^{(3)}\zeta_y^{(4)} = u_y^{(4)}$$

It is now assumed that $\underline{\beta}^{(e)}$ and $\underline{\gamma}^{(e)}$ are continuous in the same sense as $\underline{\epsilon}^{(e)}$. This is not implied by displacement continuity, but it is expected to assure a certain degree of smoothness in the distribution of the flow and damage strains. In any event, the alternative would appear to involve computing a prohibitive number of inelastic nodal parameters.

Suppose for instance that there are M plane triangular elements with a total of N nodes. Under the present assumption there are 2N values each of $\underline{\beta}^{(e)}$ and $\underline{\gamma}^{(e)}$ to compute. But otherwise there would be 6M values of $\underline{\beta}^{(e)}$ and $\underline{\gamma}^{(e)}$ to determine, and M is nearly twice N. Evidently, the inelastic smoothness assumption is very convenient in regard to computational effort.

Referring to Equation 21 and Figure 2, continuity of $\underline{\beta}^{(e)}$ is expressed as follows:

$$
\begin{aligned}
&{}^{(1)}\beta_x^{(1)} = {}^{(2)}\beta_x^{(1)} \qquad && {}^{(1)}\beta_y^{(1)} = {}^{(2)}\beta_y^{(1)} \\
&{}^{(2)}\beta_x^{(2)} = {}^{(3)}\beta_x^{(2)} \qquad && {}^{(2)}\beta_y^{(2)} = {}^{(3)}\beta_y^{(2)} \\
&{}^{(1)}\beta_x^{(4)} = {}^{(2)}\beta_x^{(4)} = {}^{(3)}\beta_x^{(4)} \\
&{}^{(1)}\beta_y^{(4)} = {}^{(2)}\beta_y^{(4)} = {}^{(3)}\beta_y^{(4)}
\end{aligned} \tag{22}
$$

For $\underline{\gamma}^{(e)}$, Equation 22 holds with gamma substituted everywhere for beta.

Note, however, that Equation 12a,c must now be modified for consistency with the nodal continuity of $\underline{\beta}^{(e)}$ and $\underline{\gamma}^{(e)}$. Repeating Equations 12a,c in updated notation

$$\dot{\underline{\beta}}^{(e)} = \underline{z}_f^{(e)} \left(\underline{\sigma}^{(e)}, \underline{\beta}^{e}, k_f^{(e)}\right) \tag{23a}$$

$$\dot{\underline{\gamma}}^{(e)} = \underline{z}_d^{(e)} \left(\underline{\sigma}^{(e)}, \underline{\gamma}^{(e)}, k_d^{(e)}\right) \tag{23b}$$

A satisfactory modification is to replace Equations 23a,b with:

$$^{(e)}\dot{\varepsilon}_m^{(n)} = \frac{1}{e_n} \sum_e {}^{(e)}z_{f_m}^{(n)} \left(\underline{\zeta}^{(e)}, \underline{\beta}^{(e)}, k_f^{(e)}\right) \qquad (24a)$$

$$^{(e)}\dot{\gamma}_m^{(n)} = \frac{1}{e_n} \sum_e {}^{(e)}z_{d_m}^{(n)} \left(\underline{\zeta}^{(e)}, \underline{\gamma}^{(e)}, k^{(e)}\right) \qquad (24b)$$

where e_n is the number of elements sharing the n^{th} node.

We now state the modified relations holding at the fourth node in Figure 2. First define $\zeta_y^{(4)}$ by

$$\zeta_y^{(4)} = \frac{1}{3}\left({}^{(1)}\zeta_y^{(4)} + {}^{(2)}\zeta_y^{(4)} + {}^{(3)}\zeta_y^{(4)}\right)$$

and by virtue of displacement continuity

$$\zeta_y^{(4)} = {}^{(1)}\zeta_y^{(4)} = {}^{(2)}\zeta_y^{(4)} = {}^{(3)}\zeta_y^{(4)} .$$

The quantities $\beta_y^{(4)}$ and $\gamma_y^{(4)}$ are analogously defined.

Using Equation 22 together with the constitutive model represented by Equations 12a-c, it follows that

$$\dot{\beta}_y^{(4)} = \frac{1}{3}\left(\sum_{e=1}^{3} \nu_f^{(3)}\right) \psi_y^{(4)} \qquad (25a)$$

where

$$\psi_y^{(4)} = 2\mu\, \zeta_y^{(4)} - (2\mu + c_f)\, \beta_y^{(4)}$$

$$\underline{\psi}^{(e)} = 2\mu\, \underline{\zeta}^{(e)} - (2\mu + c_f)\, \underline{\beta}^{(e)}$$

$$\nu_f^{(e)} = \left[2\mu\, A^{(e)}\right]^{-1/2} \left[(\underline{\psi}^{(e)})^H\, K_f^{(e)}\, \underline{\psi}^{(e)}\right]^{1/2}$$

and $A^{(e)}$ is the area of the e^{th} element.

For damage the corresponding relations are

$$\dot{\gamma}_y^{(4)} = \frac{1}{3}\left(\sum_{e=1}^{3} \nu_d^{(e)}\right) \chi_y^{(4)}$$

in which

$$\chi_y^{(4)} = \kappa\, \zeta_y^{(4)} - (\kappa + c_d)\, \gamma_y^{(4)}$$

$$\underline{\chi}^{(e)} = \kappa\, \underline{\zeta}^{(e)} - (\kappa + c_d)\, \underline{\gamma}^{(e)}$$

$$\nu_d^{(e)} = \left[3\kappa A^{(e)}\right]^{-1/2} \left[\left\{\underline{\chi}^{(e)}\right\}^H K_d^{(e)}\, \underline{\chi}^{(e)}\right]^{1/2} .$$

Finally, we consider the nodal balance of forces. The external force $\underline{P}^{(e)}$ applied to the e^{th} element is balanced by the equivalent reaction force $\underline{Q}^{(e)}$ consisting of inertial, elastic, flow and damage parts:

$$\underline{Q}^{(e)} = \underline{Q}_I^{(e)} + \underline{Q}_E^{(e)} + \underline{Q}_F^{(e)} + \underline{Q}_D^{(e)}$$

and

$$\underline{Q}_I^{(e)} = M^{(e)}\, \ddot{\underline{\zeta}}^{(e)} \qquad \underline{Q}_E^{(e)} = (K_f^{(e)} + K_d^{(e)})\, \underline{\zeta}^{(e)}$$

$$\underline{Q}_F^{(e)} = -\, K_f^{(e)}\, \underline{\beta}^{(e)} \qquad \underline{Q}_D^{(e)} = -\, K_d^{(e)}\, \underline{\gamma}^{(e)}$$

Clearly, equilibrium of an element requires that

$$\underline{P}^{(e)} = \underline{Q}^{(e)} .$$

We now illustrate nodal force balance using Figure 2, for which

$$\underline{Q}^{(1)} = \{{}^{(1)}q_x^{(1)} \quad {}^{(1)}q_y^{(1)} \quad {}^{(1)}q_x^{(4)} \quad {}^{(1)}q_y^{(4)} \quad {}^{(1)}q_x^{(5)} \quad {}^{(1)}q_y^{(5)}\}^H$$

$$\underline{Q}^{(2)} = \{{}^{(2)}q_x^{(1)} \quad {}^{(2)}q_y^{(1)} \quad {}^{(2)}q_x^{(2)} \quad {}^{(2)}q_y^{(2)} \quad {}^{(2)}q_x^{(4)} \quad {}^{(2)}q_y^{(4)}\}^H$$

$$\underline{Q}^{(3)} = \{{}^{(3)}q_x^{(2)} \quad {}^{(3)}q_y^{(2)} \quad {}^{(3)}q_x^{(3)} \quad {}^{(3)}q_y^{(3)} \quad {}^{(3)}q_x^{(4)} \quad {}^{(3)}q_y^{(4)}\}^H .$$

To balance the external force acting vertically at the fourth node in Figure 2,

$$P = {}^{(1)}q_y^{(4)} + {}^{(2)}q_y^{(4)} + {}^{(3)}q_y^{(4)} \qquad (26)$$

Simply stated, the sum of the equivalent reaction forces contributed by the elements sharing a node is set equal to the external force applied at the node. Equation (26) is readily extended to more general situations.

The constitutive relations, for example Equation 25 a,b, together with the nodal force balance relations such as Equation 26 comprise a system of ordinary differential equations in time. Under suitable initial conditions, the equations may be integrated numerically to furnish the nodal displacement, flow and damage parameters as functions of time.

CONCLUSION

The finite element method has been applied to a constitutive model describing flow and damage in rapidly loaded structural materials. A system of ordinary differential equations in time has been obtained for nodal displacement, flow and damage parameters. The formulation is "consistent" in that the inelastic strain approximants involve the same interpolation operators as the corresponding parts of the total strain approximant. Certain interelement continuity conditions are imposed on the flow and damage strains. Numerical results will be reported in a subsequent article.

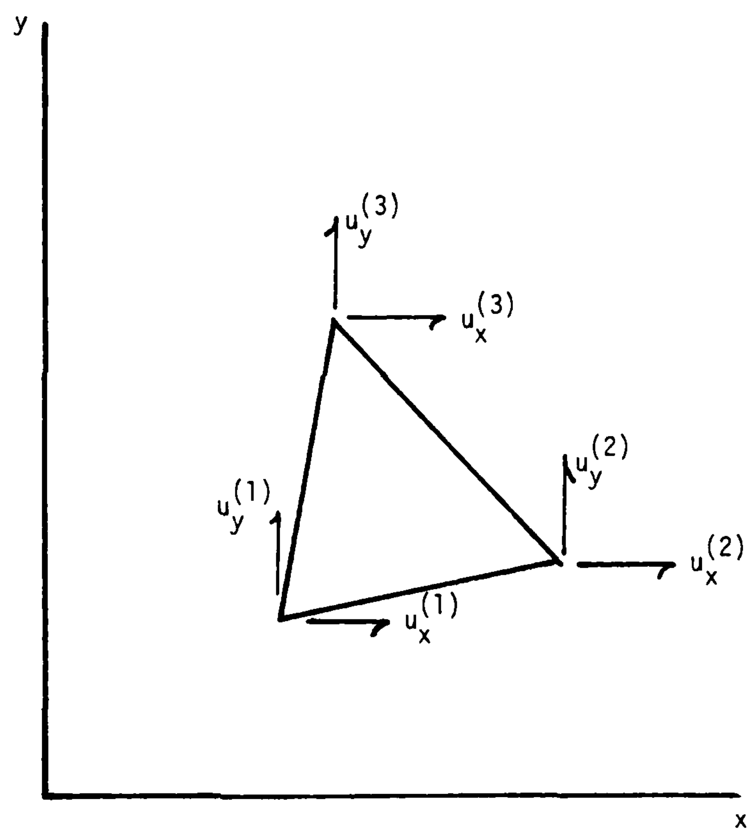

FIGURE 1 TRIANGULAR ELEMENT

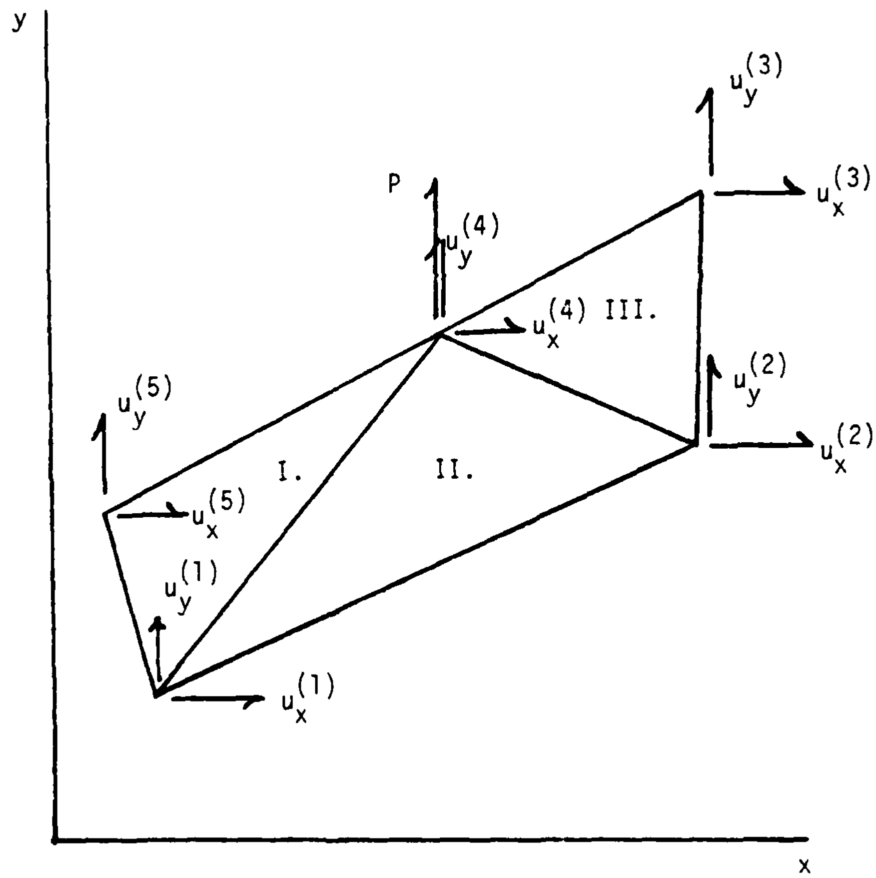

FIGURE 2 ASSEMBLAGE OF TRIANGULAR ELEMENTS

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